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1999 J. Phys. A: Math. Gen. 32 4419

(http://iopscience.iop.org/0305-4470/32/24/307)

J. Phys. A: Math. Gen. **32** (1999) 4419–4432. Printed in the UK PII: S0305-4470(99)99786-X

Classical and non-classical interaction of kinks in some bubbly mediums

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Received 30 November 1998

Abstract. Properties of nonlinear waves in one of the nonlinear partial differential equations are investigated. The kinks with the solitonic, but unusual from the classic viewpoint, behaviour are found. The equation itself does not possess the Painlevé property.

1. Introduction and preliminary

Since the discovery of soliton interaction by Zabusky and Kruskal [1] and the invention of the inverse scattering transform (IST) by Gardner *et al* [2] at the end of the 1960s nonlinear partial differential equations have been extensively studied in mathematics and physics. Much effort has been made to understand their properties, and much progress has been achieved up to now. However, in reality, only so-called integrable systems have been investigated so far. A number of methods and structures are applicable and associated with them: IST, Bäcklund transformations, master-symmetries, infinite hierarchies of conservation laws, etc. Furthermore, though such systems usually imply some simplification, they do play an important role in many applications from traditional hydrodynamics or optics to biology and neurophysiology. For these reasons, the overwhelming majority of research work deals with, namely, 'integrable' equations despite the fact that their behaviour is very similar, and their main features (e.g., a phase shift or, say, a Hirota's ansatz) are well known and widely described in the literature.

In contrast, 'nonintegrable' models are a considerably less studied area of nonlinear science. Any theory is still absent, and the remarkable fact is that they may differ radically. One major surprise is the existence of systems with solutions and behaviour analogous to classical solitonic cases [3,4], although these kind of waves (they were called 'soliton structures' in [5]) are very seldom realized in practice.

In this paper one more nonlinear partial differential equation is proposed and investigated via both numerical and analytical techniques. The equation may arise as a model for the wave processes in specific bubbly mediums [6]. It can be classified as a 'nonintegrable' one and admits solutions in the form of kinks with special properties. These structures play an important role in the system's behaviour and can be generated from appropriate initial disturbances. They are of solitonic features. To be precise, they elastically interact with other localized waves. While, as a result of a collision between two such kinks, new kinks with other wavenumbers and velocities are formed. In other words, after the interaction we not only have a phase

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shift, but also a change-over from one state to another. This switch always keeps the structure itself, however. To our knowledge, this is the only equation with such properties, which could be interpreted as 'exchange interaction'. In this respect the results obtained here are very important, because they conform with the view of the above-mentioned 'soliton-structures' proposed in [5, 7, 8].

This paper is organized as follows. First, the ground for the nonlinear equation is briefly adduced. In doing so, the derivation mainly follows the same procedure as [9, 10], but takes into account the additional assumptions relevant in our case. This is performed in section 2. In section 3, the derived equation is investigated by means of the invariant singular manifold approach from [11]. We start with a recapitulation of its theses and restrict the analysis to a brief sketch. (An excellent survey is given in [12].) Section 4 is devoted to the computer simulation for various initial data and a check of the properties of the kinks derived previously. The paper ends with some discussion and remarks in the conclusion.

2. The physical model

In [6] the existence of peculiar submicrobubbles in aqueous electrolytes was theoretically predicted, and a number of experimental data were presented, which could be treated as the evidence. Such submicrobubbles are gaps in the fluid structure charged because of ion hydration of the solution. Their diameter was estimated as 100–1000 Å. Here, we propose a nonlinear equation for describing pressure waves in such hypothetical mediums. The derivation mainly follows the same line as the preceding ones for traditional cases [9, 10], but some assumptions valid for the above scales and situation are made.

For simplicity the plane flow and an incompressible viscous liquid are considered. In the one-velocity approximation for the mixture, the hydrodynamic system takes the form

$$\rho_t + (\rho u)_x = 0 \tag{1}$$

$$\rho(u_t + uu_x) = -p_x + \tfrac{4}{3}\nu u_{xx} \qquad \nu > 0 \tag{2}$$

(ρ, u, p, ν are the density and velocity of the medium and the liquid pressure and viscosity, respectively.) It is closed by the Rayleigh–Plesset equation for a bubble and by the expression for the mean density

$$p + \frac{2\sigma}{R} + \rho_l\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right) + 2\nu\frac{\dot{R}}{R} - \frac{q^2}{4\pi R^4} = 0 \qquad \sigma > 0 \tag{3}$$

$$\rho = \rho_l(1 - \tfrac{4}{3}\pi R^3 n). \tag{4}$$

Where the surface tension, bubble and liquid densities are denoted by σ, n and ρ_l as well as the bubble radius and charge by R and q. In our approach the gas pressure in the gap becomes too small and is treated as negligible.

Next, let us introduce the new independent variables instead of R and ρ in (3) and (4) as follows:

$$\begin{aligned} &R = R_0 + \delta \qquad |\delta| \ll R_0 \ll 1 \\ &\rho = \rho_0 - a\delta + \mathrm{o}(\delta) \qquad a = 4\pi R_0^2 n\rho_l > 0. \end{aligned} \tag{5}$$

Obviously, R_0 and ρ_0 are the undisturbed bubble radius and mixture density. From equation (3) one concludes

$$p = p_0 - b\delta - c\dot{\delta} + \mathrm{o}(\delta). \tag{6}$$

(The dynamic terms are small enough in this case.) Here p_0 (the undisturbed pressure), b and c are used as abbreviations:

$$p_0 = \frac{q^2}{4\pi R_0^4} - \frac{2\sigma}{R_0}$$
$$b = \frac{q^2}{\pi R_0^5} - \frac{2\sigma}{R_0^2}$$
$$c = \frac{2\nu}{R_0} > 0.$$

It can be easily checked that $b > 0$ for physically relevant values of p_0 and R_0, because the expression for it is written down as

$$b = \frac{3q^2}{4\pi R_0^5} + \frac{p_0}{R_0}.$$

Substituting (5) into (1), in the lowest order one arrives at the equation

$$a\rho_t - \rho_0 u_x = 0. \tag{7}$$

This is clear to be valid for u of order unity only for long enough waves. The latter confirms the natural assumption that the characteristic length of the perturbation is much greater than the distance between bubbles, and it is not possible to particularize their distribution. After that, taking into account (5) and (6), and retaining the leading terms, equation (2) is replaced by

$$\rho_0(u_t + uu_x) = b\delta_x + c\delta_{tx} + \tfrac{4}{3}\nu u_{xx} \tag{8}$$

for $u \sim \mathrm{O}(1)$, sufficiently long waves and the considerable viscosity.

One can eventually rearrange (7) and (8) into the system for δ

$$\begin{aligned} \delta_t &= \frac{\rho_0}{a} u_x \\ \delta_x &= \frac{\rho_0}{b}(u_t + uu_x) - \left(\frac{c\rho_0}{ba} + \frac{4\nu}{3b}\right) u_{xx}. \end{aligned} \tag{9}$$

Moreover, its compatibility condition, after scaling as

$$u \longrightarrow \pm 2\left(\sqrt{\frac{b}{a}}\right) u$$
$$t \longrightarrow \frac{a}{b}\left(\frac{c}{a} + \frac{4\nu}{3\rho_0}\right) t$$
$$x \longrightarrow \pm\sqrt{\frac{a}{b}}\left(\frac{c}{a} + \frac{4\nu}{3\rho_0}\right) x$$

to normalize the coefficients to unity, results in

$$u_{tt} - u_{xx} + (u^2)_{tx} - u_{txx} = 0. \tag{10}$$

The latter equation is the subject of study in the following. We only need to add that since we have two small parameters $|\delta| \ll R_0 \ll 1$, the next approximation should take into account the dynamic terms from (3), so that

$$p = p_0 - b\delta - c\dot{\delta} - d\ddot{\delta} + \mathrm{O}(\delta^2) \qquad d = \rho_l R_0 > 0.$$

As a result, the equation

$$u_{tt} - u_{xx} + (u^2)_{tx} - u_{txx} - \varepsilon u_{ttxx} = \mathrm{o}(\varepsilon) \qquad 0 < \varepsilon \ll 1 \tag{11}$$

with

$$\varepsilon = \frac{db}{a^2\left(\frac{c}{a} + \frac{4\nu}{3\rho_0}\right)^2}$$

will be the perturbed version of (10).

3. The invariant Painlevé analysis and singular manifold equations

Although, as is now becoming clear, the Painlevé property (having poles as the only movable singular point, i.e. whose location depends on the initial conditions) is neither sufficient nor necessary for the integrability, and there are reasons for doubting its general validity (see [12,13] and references therein), all the classical soliton equations possess it. And, thus, a negative result may indicate another type of nonlinear dynamics for the system under consideration.

Acting in correspondence with the main scheme of the singular manifold method [12] and using the representation of the invariant Painlevé analysis due to [11], one presents u in (10) as the power series expansion

$$u(x,t) = \sum_{i=-n}^{+\infty} w_i(x,t)\cdot\chi^i(x,t) \qquad n \in N. \tag{12}$$

Where w_i are functions of the independent variables being determined in the recursive manner, and χ obeys the Riccati equations

$$\chi_x = 1 + \frac{S}{2}\chi^2 \tag{13}$$

$$\chi_t = -C + C_x\chi - \tfrac{1}{2}(CS + C_{xx})\chi^2 \tag{14}$$

with the compatibility condition to $S(x,t)$ and $C(x,t)$

$$S_t + C_{xxx} + 2SC_x + CS_x = 0. \tag{15}$$

As was shown in [11], functions $\chi = (\frac{f_x}{f} - \frac{f_{xx}}{2f_x})^{-1}$, $S = \frac{f_{xxx}}{f_x} - \frac{3}{2}(\frac{f_{xx}}{f_x})^2$ and $C = -\frac{f_t}{f_x}$ are naturally embedded into the original Painlevé analysis with the singular manifold function $f(x,t)$

$$u(x,t) = \sum_{i=-n}^{+\infty} W_i(x,t)\cdot f^i(x,t) \qquad n \in N.$$

In so doing, all arising relations can laconically be expressed in terms of them, and (13)–(15) are just the identities. S is called the Schwarzian, and C has the dimension of a velocity. For the simplest singular function $f = 1 + \exp[k(x - vt)]$ (usually corresponding to kink/soliton solutions) they are equal to $-k^2/2$ and v respectively.

Return again to (12) and substitute it into our equation and then equate expressions at different powers of χ to zero. The leading term is seen to be $-\chi^{-1}$. There is no difficulty in evaluating other w_i except at the resonances $i = 1, 2$. In these cases, the following constraints arise

$$C_t + CC_x = 0$$

$$\left(2C_x - C\frac{\partial}{\partial x}\right)(C_t + CC_x) = 0$$

that merely indicates the weak Painlevé property.

The next step that we have to consider is truncation of (12) on the constant level

$$u = -\chi^{-1} + \frac{C^2 - 1}{2C}.$$

It yields another set of equations

$$
\begin{aligned}
&C_t + CC_x = 0 \\
&C^3 C_{xxxx} + 2CC_{xx}(3C^2 C_x - 2C_x + C^2 S) + 2C^2 C_{xxx} \\
&\qquad \times (C^2 + 1) + 2C_x^3 + 3C_x S_x C^3 + C^4 S_{xx} = 0
\end{aligned}
$$

which together with (15) reduce to the simple relations to S and C via differential Gröbner bases [14]

$$
\begin{aligned}
&S_{xx} = 0 \\
&S_t + CS_x = 0 \\
&C_x = 0 \\
&C_t = 0
\end{aligned}
$$

with the trivial solutions

$$
\begin{aligned}
&S = 2c_1(x - vt) + 2c_0 \\
&C = v \qquad c_0, c_1, v = \text{const.}
\end{aligned}
$$

The substitution $\chi = \psi/\psi_x$ transforms the Riccati system (13), (14) to the linear one

$$
\begin{aligned}
&\psi_{xx} + \frac{S}{2}\psi = 0 \\
&\psi_t + C\psi_x - \frac{C_x}{2}\psi = 0.
\end{aligned}
$$

For $c_1 \neq 0$, $\psi(x, t)$ is expressed in terms of the Airy functions [15]

$$
\psi = c_2 \mathrm{Ai}\left[\frac{c_1(x - vt) + c_0}{-c_1^{2/3}}\right] + c_3 \mathrm{Bi}\left[\frac{c_1(x - vt) + c_0}{-c_1^{2/3}}\right] \qquad c_2, c_3 = \text{const.}
$$

However, the final solutions are unbounded in this case. While for $c_1 = 0$

$$
\psi = c_2 \exp\left[\frac{k}{2}(x - vt)\right] + c_3 \exp\left[-\frac{k}{2}(x - vt)\right]
$$

$(c_0 = -k^2/4)$, and finally we obtain the simplest bounded solutions of the kink-type $(c_2/c_3 = \mathrm{e}^{\varphi})$ for (10)

$$
u_{\text{kink}}(x, t) = \frac{k}{2}\left[\frac{1 - \exp(k(x - vt) + \varphi)}{1 + \exp(k(x - vt) + \varphi)}\right] + \frac{v^2 - 1}{2v} \qquad k, v, \varphi = \text{const.} \tag{16}
$$

Their interactions with other waves and with each other are numerically investigated in the next section.

The singular manifold method can also be applied to perturbed equations to perform the Painlevé analysis and perhaps find corrections to non-perturbed solutions [16] (see [17] as well). In our case, however, the introduction of εu_{ttxx} in (11) dramatically changes the situation. The first step already (the dominant behaviour) shows that such solutions cannot be represented by Laurent series as before. This may, for example, indicate the appearance of essential singularities and therefore require very serious analysis.

4. Computer simulation

In order to investigate the properties of the kinks obtained previously and the evolution of other nonlinear waves, equation (10) was numerically studied by finite-difference methods. In our computations, the following implicit scheme:

$$\begin{aligned}
&\frac{u_i^{n+1}-2u_i^n+u_i^{n-1}}{\tau^2}-\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{h^2}+\frac{u_i^n}{\tau}\left(\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2h}-\frac{u_{i+1}^{n-1}-u_{i-1}^{n-1}}{2h}\right)\\
&\qquad+2\left(\frac{u_i^{n+1}-u_i^{n-1}}{2\tau}\right)\left(\frac{u_{i+1}^n-u_{i-1}^n}{2h}\right)\\
&\qquad-\frac{1}{2\tau}\left(\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{h^2}-\frac{u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1}}{h^2}\right)=0\\
&\qquad n=\overline{0,N}\qquad i=\overline{0,I}
\end{aligned}\tag{17}$$

was employed with an appropriate choice of an initial disturbance and the boundary conditions $u_0^n, u_I^n = \text{const}$. Here, obviously, u_i^n is the mesh function at $x = hi$ and $t = \tau n$; h and τ are the grid spacings.

The difference equation (17) reduces to the set of algebraic equations with a band matrix which is simple for solving

$$a_i^n u_{i-1}^n + b_i^n u_i^n + c_i^n u_{i+1}^n = 0.$$

The above scheme was found to give the best result among others, both implicit and explicit ones, with various approximations for the nonlinear addent in (10). (The schemes for (9) were also considered.) It is of second-order accuracy $\mathrm{O}(\tau^2) + \mathrm{O}(h^2)$ and is while very economic, efficient and simple for realization. And, though it is not absolutely stable ($\tau < 1.1h$ is needed), in practice this does not constitute any difficulties.

In order to verify the real accuracy of the calculations, a number of experiments were performed with both the test solution (16) and with various time and space intervals. The results were compared with analytical ones and each other, and the errors are in good agreement with the theoretical prediction. The ration $h = 2\tau$ would be the best choice from many points of view. Moreover, there is no indication of any numerical instability in those and other computations because of round-off errors. The errors do not increase with the lapse of time and remain within the scheme accuracy after a sufficiently long period of time, although we used up to 10 000 steps in time and points on the x-axis. All computations were made in long double precision (18 figures).

In what follows some concrete results of the computer modelling are adduced. Most of them were performed with $h = 0.01$ and $t = 0.005$, which gives accurate enough results.

First of all, we investigate the interaction of the kink solution (16) with another localized wave or kink. In the former case, various situations were considered to examine solitonic properties for the solution (16). In doing so, we varied both the direction of kink moving along x-axis, its wavenumber, and the form and type of perturbation. One of such collisions is illustrated by figure 1, where the waveforms are presented versus time. The only visible effect of the kink is the slight phase shift. In opposition to this, in the latter case, the wavenumbers and velocities of the kinks are also changed after a collision, although in both cases the kinks always keep their structures, and none of them lose their identity after overlapping. This is demonstrated in figures 2 and 3, for example. Figure 2 depicts one interesting instance of such a kink–kink collision. The fact is that, in a general case, the link between the kink parameters before and after an interaction cannot be determined. However, the problem is simplified in

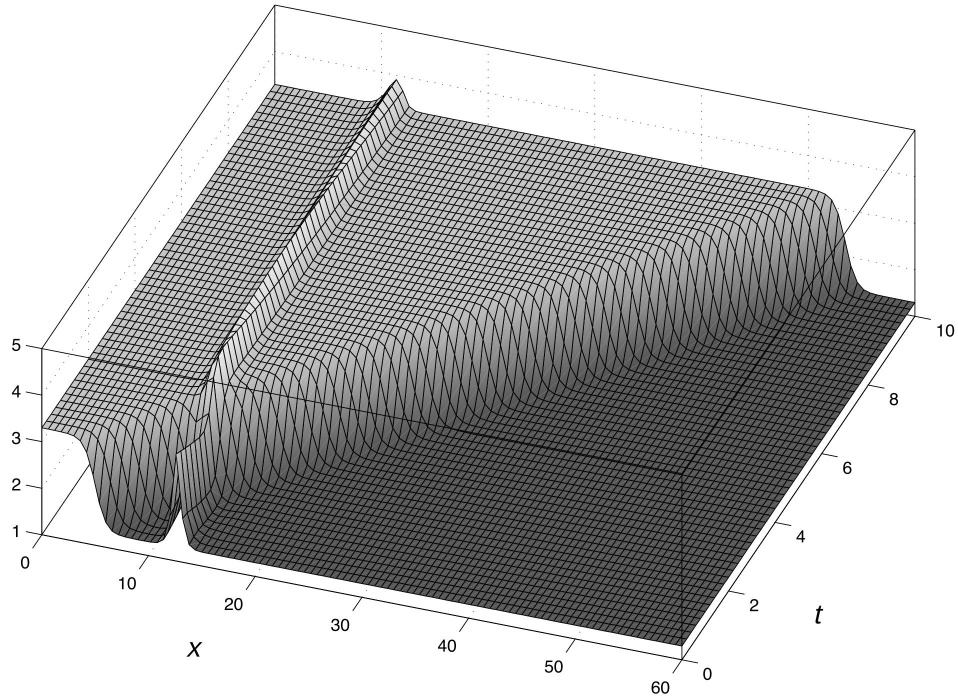

Figure 1. Interaction of the kink ($k = 2$, $v = 4.8$) with the localized perturbation.

that rather restrictive case. Here, from the boundary conditions and the symmetry we have the relation

$$\frac{v_{\text{before}}^2 - 1}{v_{\text{before}}} = \frac{v_{\text{after}}^2 - 1}{v_{\text{after}}}.$$

This enables one to interpret the process as switching from one state to another, which may be important for some technological applications.

One more circumstance should also be pointed out: the smaller the wavenumber of a kink generated after an interaction, the larger the time for its final formation; and this time grows exponentially for small wavenumbers.

The next series of the experiments were devoted to the evolution of wavefronts in the medium, and the initial data were given by the expression

$$u(x, 0) = c_1 \frac{\exp kx}{1 + \exp kx} + c_0 \qquad c_0, c_1, k = \text{const.} \tag{18}$$

with some initial velocity $v = -\frac{\partial u}{\partial t} / \frac{\partial u}{\partial x}\Big|_{t=0}$. Here the key observation is that for both fronts with $c_1 k < 0$ and for $c_1 k > 0$ there exist three distinct regions of v separated by the critical points v_- and v_+. This takes place irrespective of the parameters in (18), although v_- and v_+ depend on c_1 and c_0. Within each of the regions a scenario is qualitatively the same: the wave splits into two new ones. Their type depends on the initial perturbation type and the above region. On the other hand, there are just two types: A — the above kink solution (16) and B — a quasi-steady structure. At the critical values v_- and v_+, only one wave (and, maybe, some noise quickly dissipating) forms as a result. All this could be illustrated by the diagrams

$$\text{wavefront with} \quad c_1 k < 0;\ v > v_+ \longrightarrow B + A \tag{a}$$

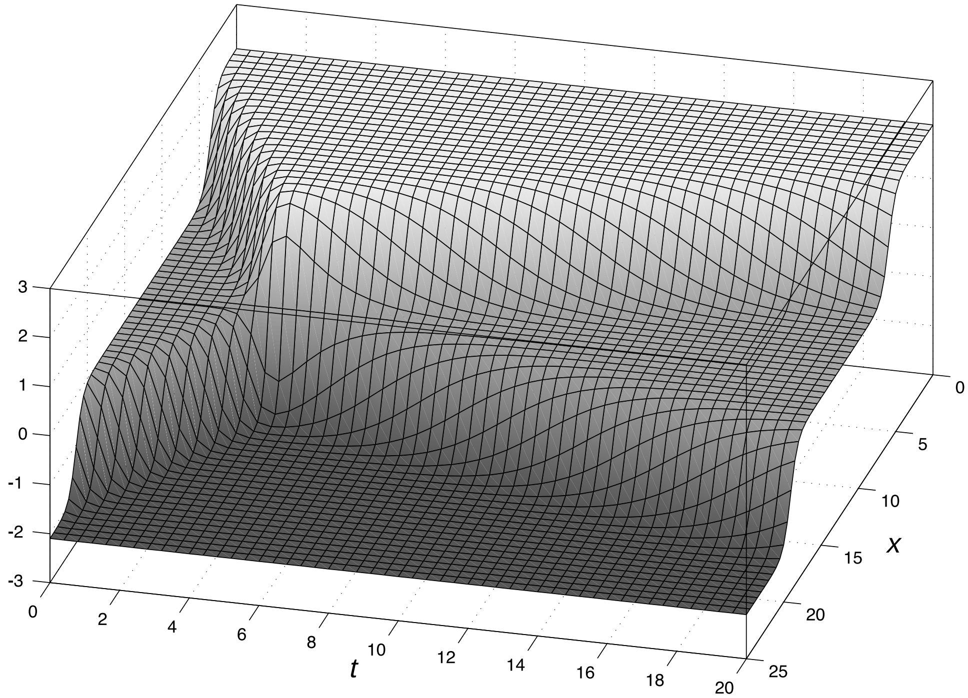

Figure 2. Kink–kink interaction ($k_1 = 2.1$, $v_1 = 2.5$, $k_2 = 2.1$, $v_2 = -2.5$).

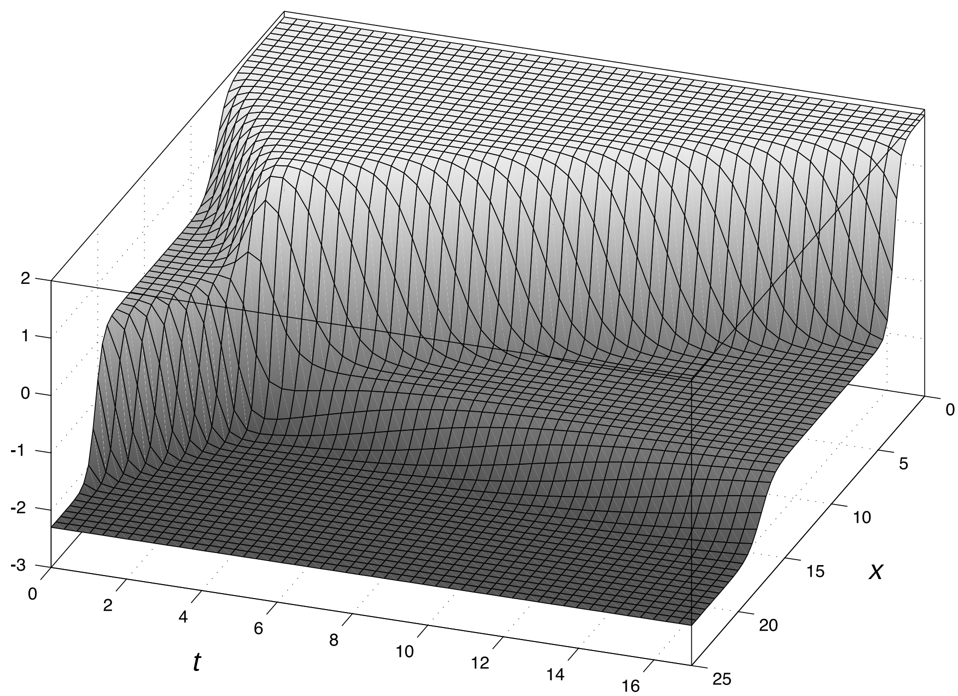

Figure 3. Kink–kink interaction ($k_1 = 1.7$, $v_1 = 2.5$, $k_2 = 2.5$, $v_2 = -2.5$).

$$\text{wavefront with} \quad c_1 k < 0;\ v = v_+ \longrightarrow A \tag{b}$$
$$\text{wavefront with} \quad c_1 k < 0;\ v_- < v < v_+ \longrightarrow A + A \tag{c}$$
$$\text{wavefront with} \quad c_1 k < 0;\ v = v_- \longrightarrow A \tag{d}$$
$$\text{wavefront with} \quad c_1 k < 0;\ v < v_- \longrightarrow A + B \tag{e}$$
$$\text{wavefront with} \quad c_1 k > 0;\ v > v_+ \longrightarrow A + B \tag{f}$$
$$\text{wavefront with} \quad c_1 k > 0;\ v = v_+ \longrightarrow B \tag{g}$$
$$\text{wavefront with} \quad c_1 k > 0;\ v_- < v < v_+ \longrightarrow B + B \tag{h}$$
$$\text{wavefront with} \quad c_1 k > 0;\ v = v_- \longrightarrow B \tag{i}$$
$$\text{wavefront with} \quad c_1 k > 0;\ v < v_- \longrightarrow B + A. \tag{j}$$

Some of them are shown in figures 4–7. It is clear that kink–kink formation considered before (figures 2 and 3) and the kinks themselves could fall into the categories (c) and (b),(d), respectively.

Finally, it is necessary to note that the wavenumber in (18) has no effect on evolution or, more precisely, plays a minimal role and only during some short initial period. Instead, the Heaviside step function could be used without loss of generality.

Accuracy of the above results is high enough. In our calculations the error between the experimental data and the expression (16) is about 10^{-4} or 0.01%, this is in full agreement with the theoretical accuracy of the difference scheme. Figures 8(*a*)–(c) (see figures 1, 3 (the left kink) and 4 respectively) depict the difference between the numerical data and u_{kink} for every type of experiment. The availability of the peaks and their non-symmetry for large values of v/k can also be explained from the theory of difference schemes, namely by the contribution

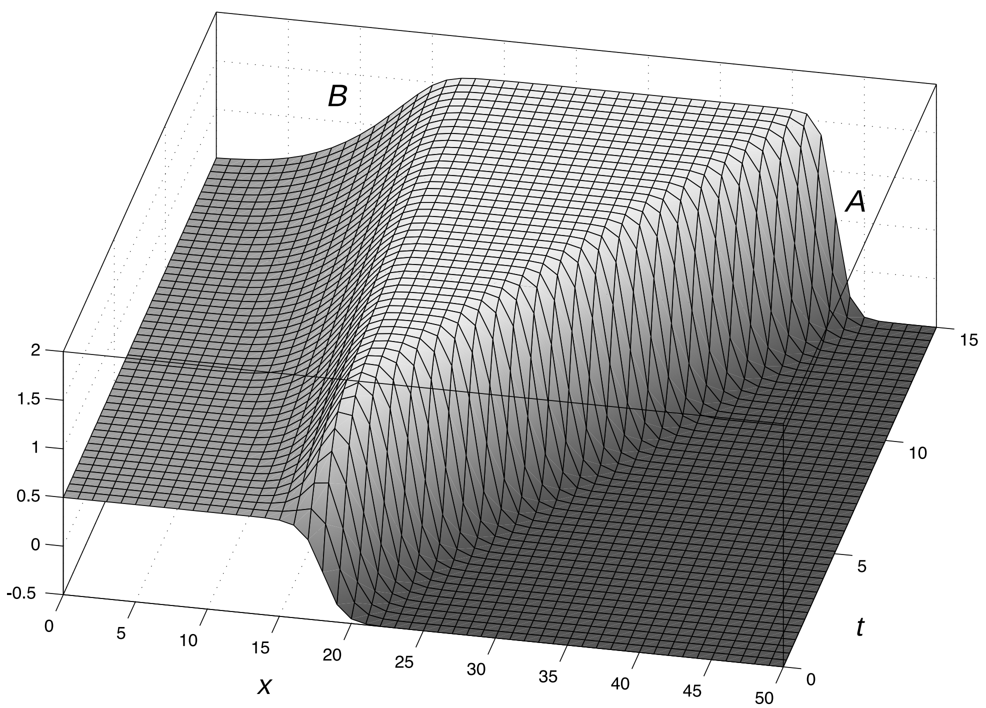

Figure 4. Evolution of the wavefront with $c_1 k < 0 (k = 1.5,\ v = 4 > v_+)$.

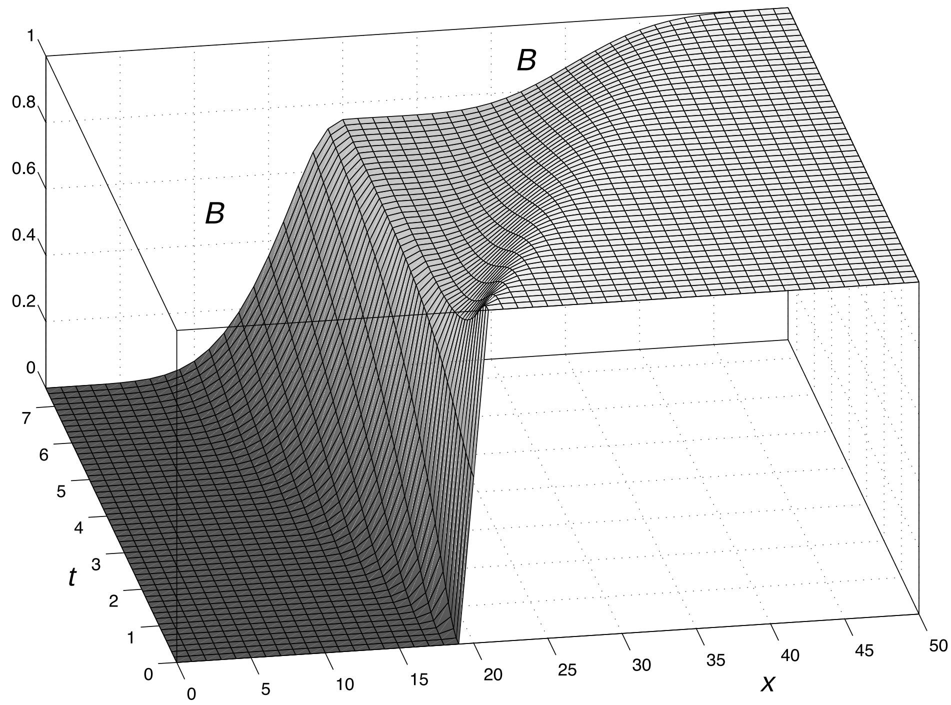

Figure 5. Evolution of the wavefront with $c_1k > 0(k = 10, v_- < v = 0 < v_+)$.

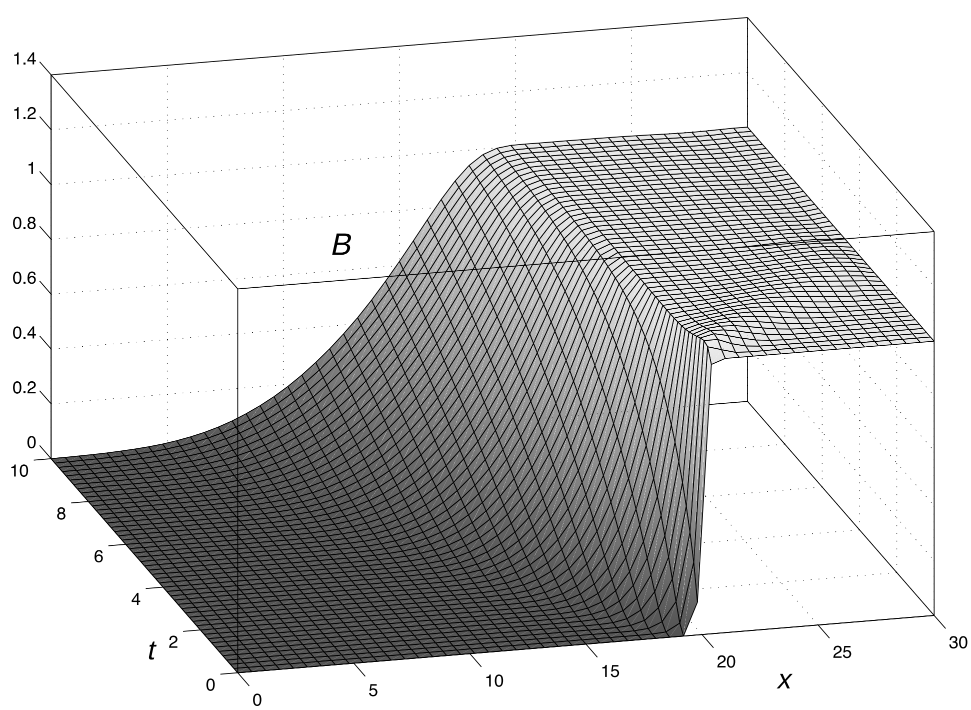

Figure 6. Evolution of the wavefront with $c_1k > 0(k = 10, v = -0.68 = v_-)$.

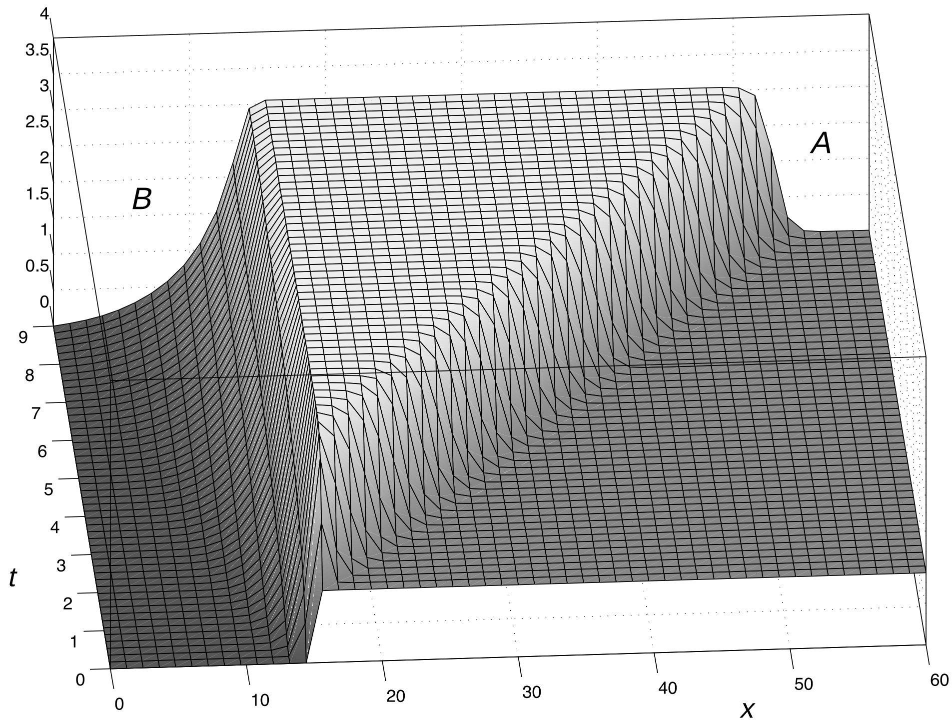

Figure 7. Evolution of the wavefront with $c_1k > 0(k = 10, v = -10 < v_-)$.

from higher spatial and time derivatives from the residual.

Collapse of a localized disturbance was also considered in our investigation. Such structures dissipate with time, perhaps after preliminary splitting into several impulses. Figure 9 exhibits such a situation.

Unfortunately, we have no possibility of verifying the solitonic properties of the above quasi-steady structure. This appears to be a rather difficult problem for computer modelling due to its specificity and the computer facility limit.

Our computer simulation has shown that the system under consideration possesses features similar to the ones of classical soliton models.

(1) It has the kink (solitonic) solutions, which elastically interact with localized disturbances.
(2) The kinks do not lose their identity when interacting with each other.
(3) They can be formed from suitable initial perturbations.

However, this model also drastically differs from them, because here the interaction is of another nature, and the kinks undergo amplitude (wavenumber) and velocity changes under mutual interactions. Such a type of interaction is novel for soliton systems, especially for $(1+1)$-dimensional scalar ones. Indeed, in such models solitons have trivial enough dynamics and only undergo a phase shift. The situation is more diverse for so-called vector solitons and $(2 + 1)$-dimensional cases. For instance, in coupled equations of the nonlinear Schrödinger family, solitons may alter their polarization (for the latest works see [18]), and two-dimensional soliton head-on collision can result in 90° scattering [19] (Ward solitons). There is one exception. To be precise, two-dimensional dromions of the Davey–Stewartson I equation change their amplitude and velocity under collisions [20]. But these are driven by nontrivial

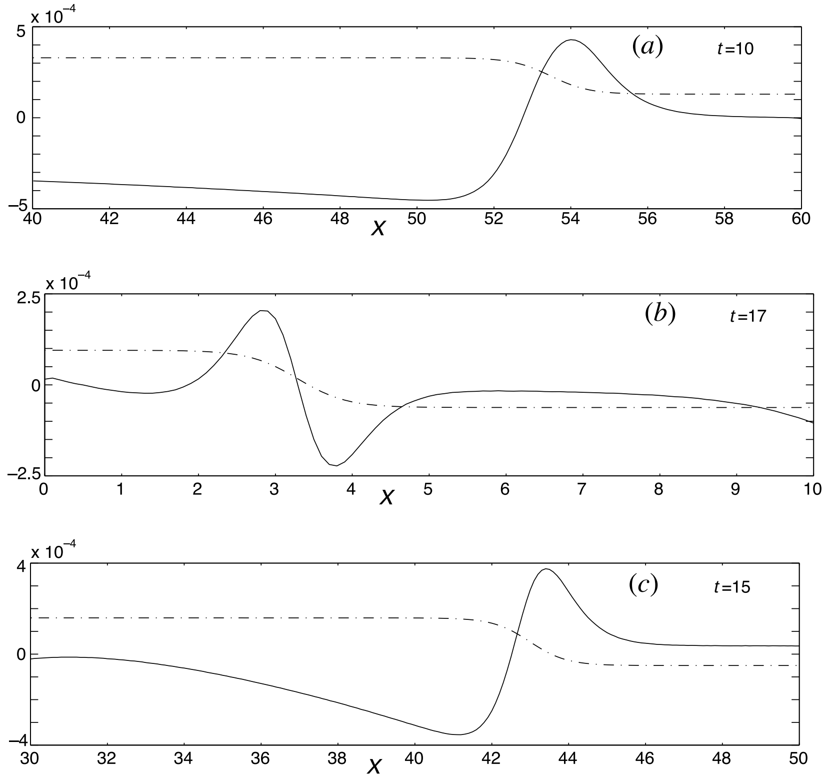

Figure 8. The difference $u_{\rm exp} - u_{\rm kink}$ between the experimental data and analytical expression (16) with some k and v. The broken curve schematically indicates the kink's profile and position. (*a*) See figure 1; $k = 2$, $v = 4.8$. (*b*) See figure 3; $k = 3.1422$, $v = 1.3817$. (*c*) See figure 4; $k = 2.0939$, $v = 1.6867$.

boundary conditions. All these types of interactions, however, fit with the traditional theory of integrable nonlinear equations, and the equations possess a full range of the related properties, including the Painlevé one, in contrast to the above equation.

While the interactions in our model are of a radically different kind from non-integrable cases as well [21,22] (see also [23]). Alone, such soliton-like solutions can be steady. However, after a collision with each other or another wave they either simply coalesce and lose their identity (some structure with complex behaviour arises) or change their form, and various defects of the envelope appear. Such a process may end with their ultimate collapse. Also, a number of second effects accompany this, such as emission of noise or new waves of a small amplitude, generation of smooth or oscillatory tails and so on. As demonstrated before, there are no signs of such side effects or deformations of the kink form in our cases.

5. Conclusion

Previously we have studied the nonlinear partial differential equation, which could arise in the theory of bubbly mediums, analytically and numerically and discovered its anomalous properties. It admits the solitonic solutions of the kink type, and they elastically interact with localized perturbations in the standard manner for solitons. But their mutual interaction is

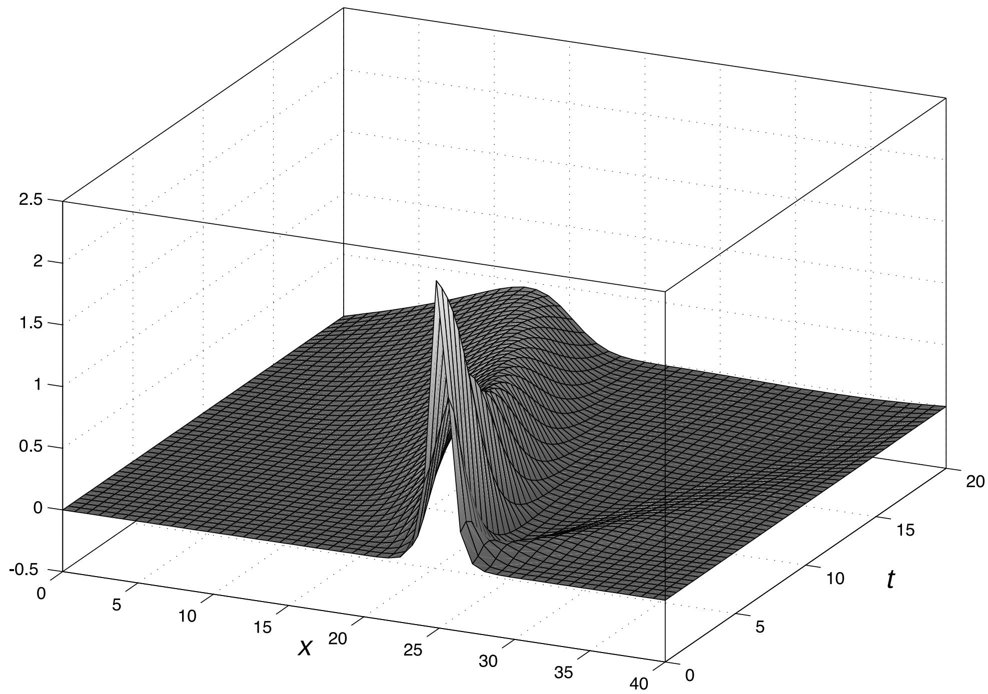

Figure 9. Dissipation of the localized perturbation.

unusual from the classical viewpoint. The equation does not possess the Painlevé properties and could be classified as a 'nonintegrable' one, though this has not been rigorously proved. Also, it seems to have no conservation laws besides itself and the following ones:

$$\frac{\partial}{\partial t}[2xu^2 + 2(t+1)u - (t^2+x^2)u_t] + \frac{\partial}{\partial x}[(t^2+x^2)(u_x - 2uu_t + u_{tx}) - 2x(u+u_t)] = 0$$

$$\frac{\partial}{\partial t}(u - tu_t) + \frac{\partial}{\partial x}(-2tuu_t + tu_x + tu_{tx}) = 0$$

$$\frac{\partial}{\partial t}(u^2 - xu_t) + \frac{\partial}{\partial x}(xu_x - 2xuu_t - u - u_t + xu_{tx}) = 0.$$

And its Lie-point symmetries are trivial and correspond just to x- and t-translations.

Besides its importance for the theory of nonlinear equations the model under consideration may be interesting from the applied viewpoint. Essentially, we have a system with unique properties for communication systems and elements of digital/analogue computers. On the one hand, the kinks are stable with respect to localized disturbances (noises), and such signals can propagate without visible damping for a long period of time. On the other hand, they can be controlled by another kink, and on this basis a nonlinear amplifier can be designed. In so doing, the second kink plays the role of a pump pulse or pilot signal. The similar property of switching from one state to another with different parameters is also interesting from the viewpoint of arithmetic and logic elements for computers. In this context investigations of similar soliton equations (in contrast to the nonlinear Schrödinger family traditional for optical and other mediums using now [24]) may be important for the development of non-semiconductor devices, e.g. biochips [25, 26], in which the silicon units would be replaced by organic molecules or genetically engineered proteins [27].

The research was initiated in [28] and was partially presented at the 6th International Conference on Evolution Equations and their Applications in Physics and Life Sciences, [29].

Acknowledgments

This work was initially supported by INTAS grant no 93-2492-ext.

The author is very grateful to Professor A Jeffrey for reading of the manuscript and encouragement and is indebted to Karlsruhe University and personally to Professor L Weis for support. He also expresses his gratitude to Professor W Neun, Dr T Wolf and Dr E Mansfield for their consultations on computer algebra. Finally, the author would like to thank Dr O Vinogradova for drawing attention to [6] and to the referees for making valuable comments and suggestions for improvement of the paper.

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